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BIOGRAPHY.

PROFESSOR WILLIAM CHAUVENET.

BY F. P. MATZ, M. SC., PH. D., NEW WINDSOR, MARYLAND.

"Professor William Chauvenet ranks among the *coryphæi* of science in America. He and Professor Benjamin Peirce have done more for the advancement of mathematical and astronomical science, and for the *raising to a higher level* of the instruction in these subjects, than any other two Americans. It is our wish, on that account, to place before the reader a somewhat full sketch of the life and works of Professor William Chauvenet."

"William Marc Chauvenet, the father of the subject of this sketch, was born at Narbonne, France, in 1790, and came to the United States in 1816. He was the youngest of four brothers, another of whom also came to this country but has left no descendants. William Marc was a man of education and culture, versed in several languages, and a constant reader. He came to America, however, in connection with a manufacturing enterprise which had its headquarters in New York, with a branch at Boston. The latter department was under Mr. Chauvenet's charge, and here he married, in 1819, Miss Mary B. Kerr, of Roxbury, Mass. This was prior to the occurrence of a heavy defalcation in the New York house, which broke up the enterprise so badly that all investments in it proved to be total losses. Mr. Chauvenet having an idea that rural life would suit his taste, bought a small farm close to *Milford, Pike County, Pennsylvania*, and it was here that his only child, WILLIAM CHAUVENET, was born, May 24, 1820.

By the advice of friends Mr. Chauvenet soon gave up his attempt at farming, and settled in Philadelphia, where his son grew to manhood. His rapid progress at school attracted such attention from his instructors, especially

in mathematics, that his father easily yielded to their advice, and sent him to Yale College, where he graduated in 1840, '*facile princeps*' in mathematics, and *high in standing* in all other branches. The honorary societies, 'Phi Delta Kappa' and 'Chi Delta Theta,' denoting respectively the fifteen of highest standing and the fifteen best writers of the class, each claimed him as a member.

Upon his return to his home he was, after a brief incumbency in a subordinate position, appointed professor of mathematics in the Navy. Late in 1841 he married Miss Catherine Hemple, of Philadelphia. Shortly after this he served a brief term on a United States vessel, as instructor to midshipmen, but did not go upon a foreign cruise, and was soon detailed to the 'Naval Asylum, then situated at Philadelphia. Here midshipmen were sent at that time, to receive instruction and examinations, principally in mathematics and the theory of navigation. The young professor was struck with the imperfections in the education of naval officers, and it was very largely through his efforts, aided by such influences as he could bring to bear on the matter, that a commission was appointed *to draft a plan* for a fixed 'Naval Academy,' corresponding to the Military Academy at West Point. Six naval officers constituted this commission, Professor Chauvenet being one of the number. The appointment of so young a man (he was but twenty four at the time) on a commission of such importance indicates what must have been his record, and the impression he made upon his seniors in years and rank.

The Naval Academy was formally called into existence in the year 1845, being located at Annapolis, Md. Professor Chauvenet was appointed to the chair of mathematics, and resided at the academy until his resignation from the Navy in 1859.

It was not long after this change of residence that he began to plan his work on trigonometry, which was published in 1850. Its title, 'A Treatise on Plane and Spherical Trigonometry,' partly indicated that it was not a students' class-book merely, but that it took up most of the more advanced applications of the subject. It soon assumed the position it still retains as the standard reference work in its line.

Some time before this publication, Professor Chauvenet had persuaded his father to retire from business and accept a position at the academy. He came as instructor in the French language, and remained at his post until his death in 1855.

It having been decided to erect an astronomical observatory at the academy, Professor Chauvenet was made professor of astronomy and put in charge of the observatory. As he became more and more interested in his work, the idea of his next treatise, 'Spherical and Practical Astronomy,' grew upon him, and, just previous to his resignation, had assumed such form that he issued a prospectus for its publication as a subscription work. This was never carried out.

In 1859 he was notified that his application for the professorship of mathematics at Yale College would be followed by his election to that position.

Almost simultaneously with this came a call to St. Louis, Mo., where

he was offered the same chair in the then newly established Washington University. After much deliberation he accepted the latter, and removed with his family (including at that time his mother) to St. Louis, in the fall of 1859.

Chancellor Hoyt, who was at the head of the 'Washington' at this time, died early in the 'sixties,' and Professor Chauvenet was elected to the vacancy. He still continued his duties as professor of mathematics, and also *resumed* his work on the 'Astronomy.' The risks of publication were great, and his means did not enable him to guarantee the publishers against loss. The Civil War was in progress, and the time seemed inopportune for such an undertaking. It was to the liberality of certain friends, chiefly to the initiative of Mr. (afterward Judge) Thomas T. Gantt, of the St. Louis bar, that a guarantee fund was raised, sufficient in the opinion of the publishers to prevent any loss to them. The work, in two octavo volumes, was published in 1863.

Few works of a scientific nature, by American authors, have been received with such universal favor, by those competent to judge of its merits, as was this. Its reputation was quite as great in Europe as here, while of course it is not (as it was never intended to be) a treatise much known outside of scientific, and more especially *astronomical*, circles. Its scope, and the rigorous methods adopted, are sufficiently indicated in the author's preface. It retaines to-day its standard character, as fully as when this was first recognized by the scientific world upon its publication.

Professor Chauvenet's mother died in St. Louis, not long after the appearance of the Astronomy, and it was but a few months later that the first symptoms of the disease that proved finally fatal to him, made their appearance. Partial recovery and resumption of his duties was followed by a long period of alternating hopes and fears, during which time he tried in vain different parts of the United States, from South Carolina to Minnesota. During this illness he worked at his *only* elementary publication, the 'Geometry,' which he undertook, partly because he had long thought that the popular texts of the day were marked by too strict an adherence to strictly 'Euclidian' methods, and partly because he wished to provide an income for his family, by the publication of a text for which he had reason to suppose there would be a larger sale than was possible with advanced treatises. The publication of this work shortly *antedated* his death, which occurred at St. Paul, Minn., December 13, 1870.

Professor Chauvenet left, so to speak, two distinct impressions behind him. By far the larger circle, in numbers, of those who knew him, were of those to whom his scientific attainments, though known, were as traditions merely, since they were in a field whose extent was to them only a matter of vague conjecture. To these he left the impression of a man of wide and varied culture, and keen critical taste. Probably few scientists of distinction were more keenly interested in lines outside of their own specialties. He was not only a *critic* in music, but to his latest day a *pianist* of no mean ability, always expressing a preference, in his own playing, for the works of Beethoven, which he rendered with an interpretation which never failed to excite the admiration of musicians whose execution surpassed his own. His knowledge of English

literature was extensive, but he read and re-read a few authors, at least in the latter part of his life, and his great familiarity with many of these gave point to the old adage, 'fear the man of few books,' though perhaps not in the sense in which these words were originally intended. He was a ready writer, and contributed at times, reviews, partly scientific, to various journals. His style was clear and unaffected, while, in the review of a pretentious or ignorant author, he had the gift of a delicate sarcasm, so light at times as only to be visible to one reading between the lines. For other pretenders he could drop this mask, and write with severity; but only twice in his life, to the knowledge of the present writer, did he ever do so. In addition to his more important writings, he was the author of a 'Lunar Method,' still used in the Navy, and invented a device called the 'great circle protractor,' by which the navigator is enabled (knowing his position) to lay down his course on a 'great circle' of the globe, without further calculation. This invention was purchased by the United States Government not long after the close of the Civil War.

Professor Chauvenet's scientific reputation needs little comment on the part of the present writer. He was one of a group of scientists in his own or cognate lines, who were the first to secure recognition abroad, as well as at home, for the position of the exact sciences in the United States. Among his more intimate scientific friends were Benjamin Peirce and Wolcott Gibbs (Harvard), Dr. B. A. Gould, and many others whose names are as household words in the history of scientific progress in this country. At the *formation* of the National Academy of Sciences he was one of the prominent members. But while his scientific reputation will outlast his personal memory, it is doubtful if to those who knew him, even of his scientific associates, it will ever be as present as his strong personal attractiveness, the result at once of an easy and varied culture, and of a simple dignity of character, which impressed alike his family, his friends, and his pupils. His family consisted, at the time of his death, of his wife, four sons, and a daughter."

"The only mathematical book written by Chauvenet and not mentioned in the above sketch is a little book entitled 'Binomial Theorem and Logarithms,' published in 1843 for the use of midshipmen at the Naval School, Philadelphia."

As regards the quality of Professor Chauvenet's books, Prof. T. H. Safford, of Williams College, says: "This excellent man and lucid writer was admirably adapted to promote mathematical study in this country. His father, a Frenchman of much culture, trained him very thoroughly in the knowledge of the French language, even in its niceties. They habitually corresponded in that language; and the son was enabled to study the mathematical writings of his ancestral country in a way which enabled him to reproduce in English their ease and grace of style, as well as their matter. In these respects his works are far more attractive than those of ordinary English writers; his Trigonometry is much the best work on the subject which I know of in any language;his Spherical and Practical Astronomy is frequently quoted by eminent *continental* astronomers; and his Geometry has raised the standard of our ordinary text-books, of which it is by far the best existing."

Professor Chauvenet's books, especially his Geometry and Trigonometry, have been used in the best of American schools. Recently Professor Byerly, of Harvard University, brought out an *excellent* revised edition of the Geometry. In their originality, the works of Professor Chauvenet are admirably rigorous. The methods of investigation adopted in his Astronomy are in accordance with what may be called the modern school of practical astronomy—or more distinctly, the *German* school—at the head of which stands the unrivaled BESSEL. His Trigonometry and Astronomy are the first *American* works to introduce the consideration of *the general spherical triangle*, or that in which the six parts of the triangle are not subjected to the condition that they shall each be less than 180°, but may have any values less than 360°. Also, all ambiguity as to the species of the six parts of the triangle is removed by determining the parts, when necessary, by *two* of their trigonometric functions, usually the *sine* and the *cosine*. In adopting this admirable feature—mainly due to *Gauss*, Professor Chauvenet was years in advance of the English and *other* American astronomers. A *new* and *simple* demonstration of the formula for the prediction of the transits of the inferior planets over the sun's disc, he gives; while *Lagrange's* well-known formula in this connection, he renders *more accurate* by his introduction of a consideration with respect to the compression of the earth. Taking the fundamental formulae of Bessel's theory of eclipses, he deduces new and elegant solutions—and these *quite as exact* as the Besselian ones. In so far as the distinctive treatment of the *occultations of planets* by the moon, is concerned, Professor Chauvenet stands as the illustrious pioneer. His Trigonometry is still *the* book in the United States Naval Academy; and last session we had a special class from Annapolis, making up Naval-Academy shortcomings in Chauvenet's Plane and Spherical Trigonometry. In that mountainous county of Pike, in that wilderness-county of north-eastern Pennsylvania, in that venatorial elysium and piscatorial paradise of 'The Keystone State,' Professor William Chauvenet was born—think of it, readers of the MONTHLY. We acknowledge our indebtedness to *President* Regis Chauvenet, Colorado School of Mines, and to *Professor* Florian Cajori, of Colorado College, for material used in this biographical sketch.

THE INSCRIPTION OF REGULAR POLYGONS.

By LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago, Chicago, Illinois.

CHAPTER VI.

[Concluded from the January Number.]

IV. Let $n=mr$, where, n being odd, m and r are both odd. Then $A_s-A_{m-s}-A_{m+s}+A_{2m-s}+A_{2m+s}-\ldots$

$$+(-1)^{\frac{r-1}{2}}A_{\frac{r-1}{2}m-s}+(-1)^{\frac{r-1}{2}}A_{\frac{r-1}{2}m+s}=0\ldots.(6).$$

Proof: $A_m-A_{2m}+A_{3m}-A_{4m}+\ldots-(-1)^{\frac{r-1}{2}}A_{\frac{r-1}{2}m}=1$, being chords of the regular r-gon. Multiplying both sides by A_s, we find

$A_s=A_{m-s}+A_{m+s}-A_{2m-s}-A_{2m+s}+\ldots$ By the method so often used we may prove that the r chords of (6) $A_s, -A_{m-s}, -A_{m+s}, A_{2m-s}$, etc., are the roots of $x^r-rx^{r-2}+\frac{r(r-3)}{1.2}x^{r-4}-\frac{r(r-4)(r-5)}{1.2.3}x^{r-6}+\ldots$

$$+(-1)^p\frac{r(r-p-1)(r-p-2)\ldots(r-2p+1)}{1.2.3\ldots p}x^{r-2p}+\ldots\pm rx-A_{rs}=0\ldots.(7).$$

This may be proved directly from the trigonometric formula:

$$2\cos r\theta=2^r\cos^r\theta-2^{r-2}r\cos^{r-2}\theta+2^{r-4}\frac{r(r-3)}{1.2}\cos^{r-4}\theta$$

$$-\ldots+(-1)^p 2^{r-2p}\frac{r(r-p-1)(r-p-2)\ldots(r-2p+1)}{1.2.3\ldots p}\cos^{r-2p}\theta+\ldots$$ from

which it follows that $A_s=2\cos\frac{s\pi}{mr}$ is a root of equation (7).

Also $-A_{m-s}=-2\cos\frac{(m-s)\pi}{mr}$ is a root of (7); for, θ then being $\frac{(m-s)\pi}{mr}$,

$2\cos r\theta=2\cos\frac{(m-s)\pi}{m}=-2\cos\frac{s\pi}{m}=-A_{rs}$. Similarly, $-A_{m+s}$, A_{2m-s}, A_{2m+s}, etc., are roots of (7). In another article I give a direct geometric-algebraic proof based upon the principles given above and the theory of symmetric functions.

If m is prime to r, one chord of each of the above groups (6) of r chords each is a root of our general equation (4) for a regular m-gon. For one

and only one of the subscripts s, $m-s$, $m+s$, $2m-s$, $2m+s, \ldots \frac{r-1}{2}m-s$, $\frac{r-1}{2}m+s$ is always divisible by r, as is seen by writing them in the equivalent form: $rm-s$, $m-s$, $(r-1)\,m-s, 2m-s, (r-2)\,m-s, \ldots \frac{r-1}{2}\,m-s$, $\frac{r+1}{2}\,m-s$, respectively. The remaining $r-1$ chords will be determined by equations whose degrees are given by the prime factors of $r-1$, — a chain of equations in which the coefficients of any one are linear functions of the roots of the preceding and of the roots of (4) for the m-gon. Hence, if the $\frac{m-1}{2}$ chords of the regular m-gon be found, we determine all the chords of the regular rm-gon by solving a series of equations whose degrees are the prime factors of $r-1$.

However, if m is divisible by r, the r chords in any of the above groups are all, or not one of them, roots of (4); for the subscripts s, $m-s$, $m+s$, $2m-s$, $2m+s, \ldots$ are all or not one divisible by r, according as s is or is not divisible by r. Hence, by the grouping of the $\frac{n-1}{2}$ chords of the n-gon into $\frac{m-1}{2}$ groups of r chords each, we can not lower or avoid equations of the rth degree of the form (7). More definitely, if r be a prime number and if m be divisible by r, we must, for any grouping whatever of the chords of the rm-gon, solve one or more equations of degree r.

The regular polygon of mr sides depends for inscription, if m be prime to r, upon the same equations as does the regular m-gon, together with equations whose degrees are the prime factors of r—1; but, if m contains as factor the prime number r, upon an equation of degree r and of the form (7), *in addition to those required by the regular m-gon.*

To inscribe a regular polygon of $n = a^{\alpha} b^{\beta} c^{\gamma} \ldots$ sides, therefore, where a, b, $c, \ldots$ are *different* prime numbers, it is necessary to solve $\alpha-1$ equations of degree a, $\beta-1$ of degree b, etc., besides equations whose degrees are given as the prime factors of $\frac{a-1}{2}, \frac{b-1}{2}, \frac{c-1}{2}, \ldots$

It follows that the regular $(2^x+1)m$-gon depends for inscription upon the same equations as the regular m-gon, provided 2^x+1 be a prime number, and is inscriptible if the latter is. Hence, a regular polygon of $(2^x+1)(2^y+1)(2^z+1)\ldots$ sides, where the factors are *different* prime numbers, is geometrically inscriptible. We thus have Gauss' theorem:

A regular polygon the number of whose sides is a prime number of the form 2^x+1, or the product of two or more different primes of that form, or a power of 2 times such an expression, is geometrically inscriptible; and inversely.

Of the regular polygons with less than 200 sides, 31 are geometrically inscriptible:

3,4,5,6,8,10,12,15,16,17,20,24,30,32,34,40,48,51,60,64,68,80,85,96, 102, 120,128,136,160,170,192;

67 depend for inscription upon *cubics* only: 7,9,13,14,18,19,21,26,27, 28,35,36,37,38,39,42,45,52,54,56,57,63,65,70,72,73,74,76,78,81,84,90, 91, 95, 97, 104,105,108,109,111,112,114,117,119,126,130,133,135,140,144,146,148,152,153, 156,162,163,168,171,180,182,185,189,190,193,194,195;

23 depend upon *quintics* only: 11,22,25,33,41,44,50,55,66,75,82,88, 100,101,110,123,125,132,150,164,165,176,187; 17 depend upon *cubics* and *quintics*: 31,61,62,77,93,99,122,124,143,151,154,155,175,181,183,186,198;

8 depend on equations of 7th degree only: 29,58,87,113,116,145,174,197;

9 depend on equations of 3rd and 7th degrees: 43,49,86,98,127,129,147,172, 196; 2 on equations of 5th and 7th degrees: 71 and 142; the 40 remaining depend on equations of the 11th or higher degrees.

An idea of the comparative infrequency of the geometrically inscriptible regular polygons if we advance to large numbers is found in the fact (as shown by a complete table I have made) that there are only 206 of them with sides less than a million.

In a paper*on *The Number of Inscriptible Regular Polygons*, I proved that between the successive powers of 2 lie 1,2,3,4,5, etc. numbers giving inscriptible regular polygons; thus

[3],4,[5,6],8,[10,12,15],16,[17,20,24,30],32,[34,40,48,51,60],64,....

or generally, n such numbers lie between 2^n and 2^{n+1}, for every value of $n<32$. For $n \overline{\overline{>}} 32$ but <128, 31 such numbers lie between 2^n and 2^{n+1}. For values of $n \overline{\overline{>}} 128$, there would be 31 such numbers in each interval *if* (as is not yet known) $2^{128}+1$ is a composite number; but $31+l$ such numbers between 2^{128+l} and 2^{129+l}, if it is a prime number, l being <128.

Hence the number of inscriptible regular polygons below 2^x+1 sides, for $x<32$, is $\frac{1}{2}(x-1)(x+2)$; for $x \overline{\overline{>}} 32$, but <128, is $(32x-497)$.

In conclusion, I will add a direct geometric proof of our fundamental theorem (5): $A_1-A_2+A_3-A_4+A_5-\ldots\ldots-(-1)^pA_p=1$.

Suppose $A_1-A_2+A_3-\ldots\ldots\pm A_p=x$. $\therefore\ xA_1=A_1(A_1-A_2+A_3-\ldots\ldots\pm A_{p-2}\pm A_{p-1}\pm A_p)=2+A_2-A_1-A_3+A_2+A_4-A_3-A_5+\ldots\ldots\pm A_{p-3}\pm A_{p-1}\pm A_{p-2}\pm A_p\pm A_{p-1}\pm A_p=2+A_1-2(A_1-A_2+A_3-\ldots\ldots\pm A_{p-2}\pm A_{p-1}\pm A_p)=2+A_1-2x$.

$\therefore\ (x-1)(2+A_1)=0.$ But $A_1+2=0$. $\therefore\ x=1$.

Errata in October Number:—p.343, line 20 add *when n is a prime number*; p. 344, line 8 read $A_s^2=2-A_{n-2s}$.

* Bulletin of the New York Mathematical Society, February, 1894.

A PROPOSITION IN REFERENCE TO CENTRE OF GRAVITY, AND ITS DEMONSTRATION.

By J. W. NICHOLSON, A. M., LL.D., President and Professor of Mathematics, Louisiana State University and Agricultural and Mechanical College, Baton Rouge, Louisiana.

PROPOSITION. The point $P'(x',y',z')$ is the centre of gravity of the mass m if the sum (s) of the squares of the distances from P' to every point of m, is a minimum.

PROOF. Let $P(x,y,z)$ be any point of m, then the square of the distance PP' is $PP'^2=(x-x')^2+(y-y')^2+(z-z')^2$,

$$\text{and } s=\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}[(x-x')^2+(y-y')^2+(z-z')^2]dxdydz.$$

Representing $\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}$ by $\int$ and $dxdydz$ by dm,

$$\text{we have } s=\int[(x-x')^2+(y-y')^2+(z-z')^2]dm.$$

Since s is a minimum with respect to the independent variables.

x',y',z', we have $\dfrac{ds}{dx'}=0$, $\dfrac{ds}{dy'}=0$, and $\dfrac{ds}{dz'}=0$: that is,

$$(1).\quad \int(x-x')dm=0, \quad \therefore \quad x'=\frac{\int x dm}{\int dm};$$

$$(2).\quad \int(y-y')dm=0, \quad \therefore \quad y'=\frac{\int x dm}{\int dm};$$

$$(3).\quad \int(z-z')dm=0, \quad \therefore \quad z'=\frac{\int z dm}{\int dm}$$

Q. E. D.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

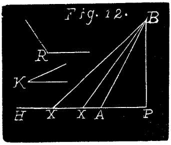
By GEORGE BRUCE HALSTED, A. M.. (Princeton); Ph. D., (Johns Hopkins), Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the January Number

SCHOLION II. In the three preceding theorems I have studiously set down this condition, that the cutting straight AP, or XA, is understood to be of a *designated length as great as you choose.*

For if, without any determinate extent of the cutting straight it be discussed precisely concerning the exhibiting and demonstrating of the concurrence of two straights at the apex of a certain triangle, whose angles at the base are given (less indeed than two right angles) as, suppose, one right, and the other less than a right by as much as two degrees, or, if you please, by less: who is so devoid of geometry that he could not immediately show the thing itself demonstratively?

For suppose (fig. 12) given any angle BAP, as, say, 88 degrees. If therefore from any point B of this AB, is let fall on the base AP (Eu. I. 12.) the perpendicular BP, it holds certainly that in this triangle ABP would be exhibited demonstratively the desired concurrence in this point B.



But if the other angle at the base is postulated, and it less than a right, as suppose 84 degrees, which indeed the given angle K represents : then (Eu. I. 23.) one would be able to make toward the parts of the straight AB an equal angle APD, PD meeting this AB in D some intermediate point of it. Wherefore the desired concourse is again obtained demonstratively in this point D.

But finally: if the other angle is postulated obtuse, but yet less than 92 degrees, lest with the other given angle BAP it should make up two rights: : this may be represented in a certain angle R of 91 degrees. It is to be shown, that there is some one point X of this AP, to which the join BX makes an angle BXA equal to the given angle R of 91 degrees: so that therefore under a certain cutting straight AX the desired meeting in the point B many be obtained.

But we may proceed thus.

PA being produced to any point H, since the external angle BAH is (Eu. I. 13.) 92 degrees, since the interior angle BAP is by hypothesis 88 degrees; and again, (Eu. I. 16.) is greater not alone than the right angle BPA, but also, for the same reason, than any obtuse angle BXA, the point X being assumed wherever you choose within this PA, and indeed always growing

greater as the point X is assumed nearer to the point A, (Eu. I. 16.) : it is an evident consequence, that between those angles, one of 90 degrees at the point P, and the other of 92 degrees in the point A, one angle BXA is found, which is 91 degrees, truly equal to the given angle R. None the less, omitting this last observation about the obtuse angle, it is necessary most diligently to take care that the difficulty of this proposition [axiom] of Euclid be fixed in this, that it asserts the meeting of two straights; especially in that part in which they make with the cutting straight two angles less than two right angles; and assuredly that it asserts the aforesaid meeting thus, *of whatever length be the assigned transversal.*

For otherwise (as I have already mentioned in the preceding scholion) I will demonstrate that general meeting solely from the admitted meeting of this sort, when one of the angles is right; and indeed, even if it be admitted not for any assignable finite transversal, but alone admitted within the limits of any assigned very small transversal.

[To be continued.]

ARITHMETIC.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

36. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

It costs $C=\$22$ to paper a room $a=18$ feet long, $b=15$ feet wide, and $c=10$ feet high, with paper (m/n)th, $=\frac{3}{4}$, of a yard wide. Find the price of the paper per roll of $R=12$ linear yards.

Solution by the PROPOSER.

Making no allowance for "matching", the number of linear yards of paper required is $Y=\frac{1}{3}[2(a+b)c+ab](n/m), =413\frac{1}{3}$; and, consequently, the number of rolls of paper required is $N=Y/R=34\frac{4}{9}$.

Hence the price of the paper per roll is

$$P=\frac{3(m/n)R}{2(a+b)c+ab} \text{ of } \$C, =\$\tfrac{99}{155}=63\tfrac{7}{8} \text{ cents.}$$

Professor J. F. W. Scheffer, P. S. Berg, and J. A. Calderhead get \$2 70 as the result. Professor John Faught and I. L. Beverage get \$1 91$\frac{19}{31}$, and Professor T. W. Palmer gets \$178$\frac{2}{31}$. These different results are due to different interpretations of the problem.

37. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

I have three jars, A, B, and C, holding respectively $a=1$, $b=3$, and $c=5$ gallons. A is empty, B is full of water, and C is full of wine. I *fill* A from B; then I *fill* up B from C and pour the contents of A into C. After *repeating* this operation how much wine is there in B? How much in C?

I. Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

After the first operation, B contains 1 gallon of wine, C contains 4 gallons. In the second operation $\frac{4}{5}$ gallons of wine are drawn from C and $\frac{1}{3}$ gallons added to C. C contains, $4-\frac{4}{5}+\frac{1}{3}=3\frac{8}{15}$ gallons of wine.

B contains, $5-3\frac{8}{15}=1\frac{7}{15}$ gallons of wine.

II. Solution by Professor T. W. PALMER, University of Alabama, I. L. BEVERAGE, Monterey, Virginia and the PROPOSER.

After the *first* operation, A is empty, B contains $(b-a)$ gallons of water and a gallons of wine, and C contains $(c-a)$ gallons of wine and a gallons of water. In so far as changes in the proportional parts of the wine are effected by the *repetition*, A is empty, B has gained from C $[(c-a)\div c]$ of a gallons, and C has gained from B *through* A $(a\div b)$ of a gallons. Hence, after the repetition, the number of gallons of wine in B is

$$G_B=\left[1+\frac{c-a}{c}-\frac{a}{b}\right] \text{ of } a=\left[2-\left(\frac{1}{b}+\frac{1}{c}\right)a\right] \text{ of } a, =1\tfrac{7}{15}.$$

Consequently the number of gallons of wine in C is

$$G_C=c-G_B=c-\left[2-\left(\frac{1}{b}+\frac{1}{c}\right)a\right] \text{ of } a, =3\tfrac{8}{15}.$$

III. Solution by Professor J. F. W. SCHEFFER, Hagerstown, Maryland.

Let, after x such operations as described, the quantity of wine in the vessel B be $F(x)$, then $b-F(x)$ will be the quantity of water in B, $c-F(x)$ the quantity of wine in C and $F(x)$ the quantity of water in C.

Repeating the operation once more, we have in vessel B,

$$\frac{b-a}{b}F(x)\text{ wine }+\frac{b-a}{b}[b-F(x)]\text{ water}+\frac{a}{c}[c-F(x)]\text{ wine }+\frac{a}{c}F(x)\text{ water;}$$

and in C, $\frac{c-a}{c}[c-F(x)]$ wine $+\frac{c-a}{c}F(x)$ water $+\frac{a}{b}F(x)$ wine $+\frac{b-F(x)}{b}$ water. $\therefore$ $\frac{b-a}{b}F(x)+\frac{a}{c}[c-F(x)]=F(x+1)$ an equation in Finite Differences. Resolving, we find $F(x)=\frac{bc}{b+c}+C\left(\frac{bc-ab-ac}{bc}\right)^x$.

For $x=0$, $F(x)=0$. $\therefore$ $C=-\frac{bc}{b+c}$; $\therefore$ $F(x)=\frac{bc}{b+c}\left[1-\left(\frac{bc-ab-ac}{bc}\right)^x\right]$.

The quantity of wine in C is, of course, $=c-F(x)$.

For $x=2$, we have $F(x)=\dfrac{a(2bc-ab-ac)}{bc}$. For the numerical values we have $F(x)=\frac{15}{8}[1-(\frac{7}{15})^x]$ and $F(2)=1\frac{7}{15}$. The value to which $F(x)$ approximates is $\dfrac{bc}{b+c}$.

38. Proposed by J. A. CALDERHEAD, B. Sc., Superintendent of Schools, Limaville, Ohio.

What must be the thickness of a 36-inch shell, in order that it may weigh 1 ton: supposing a 13-inch shell to weigh 200 pounds, when two inches thick?

I. Solution by EDWARD R. ROBBINS, Princeton, New Jersey.

Given shell....Vol.$=\dfrac{734\pi}{3}$. Required shell of thickness $=x$....

Volume$=\dfrac{(4x^3-72x^2+1296x)\pi}{3}$. Now.... $\dfrac{734\pi}{3}:\dfrac{4(x^3-18x^2+324x)\pi}{3}::$ 1:10 which gives....$x^3-18x^2+324x-1835=0$, whence $x=7.4817$ in. the required thickness.

II. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia, and J. W. WATSON, Middle Creek, Ohio.

Volume 13-inch sphere $=\frac{1}{6}\pi 13^3=\dfrac{2197\pi}{6}$ cubic inches.

Volume of hollow$=\frac{1}{6}\pi 9^3=\dfrac{243\pi}{2}$ cubic inches.

Volume of 36-inche sphere$=\frac{1}{6}\pi 36^3=7776\pi$ cubic inches.

Volume of hollow of 36-inch sphere$=\frac{1}{6}\pi D^3$ cubic inches.

$\dfrac{2197\pi}{6}-\dfrac{243\pi}{2}=\dfrac{734\pi}{3}$ cubic inches=volume of 13-inch shell.

$\therefore$ $\dfrac{734\pi}{3}$ cubic inches weigh 200 pounds.

$\therefore$ 1 cubic inch weighs $\dfrac{600}{734\pi}=\dfrac{300}{367\pi}$ pounds.

7776π cubic inches weigh $7776\pi\times\dfrac{300}{367\pi}=\dfrac{2332800}{367}$ pounds.

$\frac{1}{6}\pi D^3$ cubic inches weigh $\frac{1}{6}\pi D^3\times\dfrac{300}{367\pi}=\dfrac{50D^3}{367}$ pounds.

But $\dfrac{2332800}{367}-\dfrac{50D^3}{367}=2000$. $\therefore$ $D^3=31976$. $\therefore$ $D=31.74$ inches.

$\therefore$ $36-31.74=4.26$, $4.26\div2=2.13$ inches.

$\therefore$ 2.13 inches is the thickness of the 36-inch shell.

PROBLEMS.

44. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

A, B, and C together bought a ship. A paid for the a/bth, $=\frac{2}{5}$th, part of the ship. B paid for the m/nth, $=\frac{3}{8}$th, part of the ship. C paid $\$M$, $=\$2000$. How many dollars did A, and B, pay?

45. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

In running a mile, A can give B $a=20$ yards; B can give C $b=88$ yards. How many yards can A give C?

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

34. Proposed by ROBERT J. ALEY, A. M., Professor of Mathematics, Indiana University, Bloomington, Indiana.

$$\Sigma_1^n \frac{(n+2)^2}{n(n+4)}=\text{what?}$$

Solution by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

$$\Sigma_1^n \frac{(n+2)^2}{n(n+4)}=\Sigma_1^n\left[1+\frac{4}{n(n+4)}\right]=\Sigma_1^n(1)+4\Sigma_1^n\frac{1}{n(n+4)}=n+4\Sigma_1^n\frac{1}{n(n+4)}.$$

$$\text{But } 4\Sigma_1^n\frac{1}{n(n+4)}=\left(\frac{1}{1}-\frac{1}{5}\right)+\left(\frac{1}{2}-\frac{1}{6}\right)+\left(\frac{1}{3}-\frac{1}{7}\right)+\left(\frac{1}{4}-\frac{1}{8}\right)+\dots$$

$$+\left(\frac{1}{n-4}-\frac{1}{n}\right)+\left(\frac{1}{n-3}-\frac{1}{n+1}\right)+\left(\frac{1}{n-2}-\frac{1}{n+2}\right)$$

$$+\left(\frac{1}{n-1}-\frac{1}{n+3}\right)+\left(\frac{1}{n}-\frac{1}{n+4}\right)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{n+1}-\frac{1}{n+2}$$

$$-\frac{1}{n+3}-\frac{1}{n+4}=\left(1-\frac{1}{n+1}\right)+\left(\frac{1}{2}-\frac{1}{n+2}\right)+\left(\frac{1}{3}-\frac{1}{n+3}\right)+\left(\frac{1}{4}-\frac{1}{n+4}\right)$$

$$=\frac{n}{n+1}+\frac{n}{2(n+2)}+\frac{n}{3(n+3)}+\frac{n}{4(n+4)}$$

$$\therefore \quad \Sigma_1^n \frac{(n+2)^2}{n(n+4)} = n\left[1+\frac{1}{n+1}+\frac{1}{2(n+2)}+\frac{1}{3(n+3)}+\frac{1}{4(n+4)}\right].$$

Also solved by *Professor G. B. M. Zerr.*

35. **Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.**

Prove that the product of two numbers, each the sum of four (4) squares may be expressed as the sum of four squares in 48 different ways and unite some or all of the 48 ways.

Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let $(a^2+b^2+c^2+d^2)(e^2+f^2+g^2+h^2)=A=$ the product of the two numbers. From Euler's Theorem we get

$$A=(ae+bf-cg-dh)^2+(af-be-ch+dg)^2$$
$$+(-ag+bh-ce+df)^2+(ah+bg+cf+de)^2,$$

$$=(-ae+bf-cg+dh)^2+(af+be+ch+dg)^2$$
$$+(ag+bh-ce-df)^2+(ah-bg-cf+de)^2,$$

$$=(ae-bf-cg+dh)^2+(af+be-ch-dg)^2$$
$$+(ag+bh+ce+df)^2+(-ah+bg-cf+de)^2,$$

$$=(ae+bf+cg+dh)^2+(-af+be-ch+dg)^2$$
$$+(ag-bh-ce+df)^2+(ah+bg-cf-de)^2,$$

$$=(ae-bf+cg+dh)^2+(af+be-ch+dg)^2$$
$$+(ag-bh-ce-df)^2+(ah+bg+cf-de)^2,$$

$$=(ae-bf-cg-dh)^2+(af+be+ch-dg)^2$$
$$+(ag-bh+ce+df)^2+(ah+bg-cf+de)^2,$$

$$=(ae+bf-cg+dh)^2+(af-be+ch+dg)^2$$
$$+(ag+bh+ce-df)^2+(ah-bg-cf-de)^2,$$

$$=(ae+bf+cg-dh)^2+(af-be-ch-dg)^2$$
$$+(ag+bh-ce+df)^2+(ah-bg+cf+de)^2.$$

The sum of four squares in eight different ways by combination of signs.

$$A=(ae+bf)-cg-dh)^2+(af-be-ch+dg)^2$$
$$+(-ag+bh-ce+df)^2+(ah+bg+cf+de)^2,$$

$$=(ag+bf-ch-de)^2+(af-bg-ce+dh)^2$$
$$+(-ah+be-cg+df)^2+(ae+bh+cf+dg)^2,$$

$$=(ah+bf-ce-dg)^2+(af-bh-cg+de)^2$$
$$+(-ae+bg-ch+df)^2+(ag+be+cf+dh)^2,$$
$$=(ae+bg-cf-dh)^2+(ag-be-ch+df)^2$$
$$+(-af+bh-ce+dg)^2+(ah+bf+cg+de)^2,$$
$$=(af+bg-ch-de)^2+(ag-bf-ce+dh)^2$$
$$+(-ah+be-cf+dg)^2+(ae+bh+cg+df)^2,$$
$$=(ah+bg-ce-df)^2+(ag-bh-cf+de)^2$$
$$+(-af+be-cg+dh)^2+(ae+bf+ch+dg)^2,$$

the sum of four squares in six different ways by combination of letters.

Since the signs of each of these six can form the sum of four squares in eight different ways, the whole number of ways is $8\times6=48$ different ways.

PROBLEMS.

46. Proposed by Professor WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics in the Ohio University, Athens, Ohio

Find θ from $\cos\theta+\cos3\theta+\cos5\theta=0$.

47. Proposed by LEONARD E. DICKSON, A. M., Fellow in Mathematics, University of Chicago, Chicago, Illinois.

Prove that $(-1)(-1)=+1$.

GEOMETRY.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

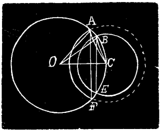
SOLUTIONS OF PROBLEMS.

34. Proposed by T. JOHN COLE, Columbus, Ohio.

A circular field contains 10 acres. A horse is tied to the fence with a rope sufficiently long to graze over one acre. Find length of the rope (1) when the horse is on the inside (2) when he is on the outside of the fence.

Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let C be the point to which the rope is fastened, A and D two points in the circumference to which the horse can graze when on the inside, B and E the points in the circumference to which the horse can graze when on the outside. Let O be the center of the given circle.



Let $OA=a=\dfrac{40}{\sqrt{\pi}}$, the radius of the given circle, and $\angle ACO=\theta$, $\angle BCO=\phi$. Then we have $AC=2a\cos\theta$, $BC=2a\cos\phi$. The area common to the two circles in the first case$=a^2(\pi+2\theta\cos2\theta-\sin2\theta)$. The area common to the two circles in the second case$=a^2(\pi+2\phi\cos2\phi-\sin2\phi)$.

Therefore the area upon which the animal can graze upon the inside of the circle is

$$a^2(\pi+2\theta\cos2\theta-\sin2\theta)=\tfrac{1}{10}\pi a^2 \ldots (1).$$

The area upon the outside is

$$4\pi a^2\cos^2\phi-a^2(\pi+2\phi\cos2\phi-\sin2\phi)=\tfrac{1}{10}\pi a^2 \ldots (1).$$

From (1) $9\pi+20\theta\cos2\theta-10\sin2\theta=0$.

From (2) $40\pi\cos^2\phi-20\phi\cos2\phi+10\sin2\phi=11\pi$.

Solving by the method of double position,

$\theta=76^\circ\ 21'\ 44''.04$,

$\phi=77^\circ\ 38'\ 25''$.

$AC=2a\cos\theta=10.64216$ rods,

$BC=2a\cos\phi=9.65892$ rods.

Good solutions to this problem were received from *J. F. W. Scheffer*, and *P. S. Berg*.

35. Proposed by LEONARD E. DICKSON, M. A., Fellow in Mathematics, The University of Chicago.

Determine the equation of lowest degree (cubic) upon which depends the inscription of the regular polygon of 37 sides.

Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Using the proposers notation as given in his excellent papers in the MONTHLY, we get for $n=37$, the following order of subscripts:

1,2,4,8,16,5,10,17,3,6,12,13,11,15,7,14,9,18.

Hence the groups are $(A_1-A_8-A_{10}-A_6+A_{11}-A_{14})=A$, $(-A_2-A_{16}+A_{17}-A_{12}+A_{15}+A_9)=B$, $(-A_4+A_5+A_3+A_{13}+A_7-A_{18})=C$.

$A+B+C=1$. $\quad AB=5(-A_1-A_8-A_{10}-A_6+A_{11}-A_{14})$
$-4(-A_2-A_{16}-A_{17}-A_{12}-A_{15}+A_9)$
$\qquad\qquad -3(-A_4+A_5+A_3+A_{13}+A_7-A_{18})$.

$\therefore\ AB=-(5A+4B+3C)=-5+B+2C$.

By symmetry, $AC=-5+A+2B$, $BC=-5+C+2A$.

$\therefore\ AB+AC+BC=-15+3(A+B+C)=-12$.

$ABC=-A\,(5-C-2A)=-A(3+2B+C)=-(3A+2AB+AC)$.

$\therefore\ ABC=-(3A-10+2B+4C-5+A+2B)=-4(A+B+C)+15$.

$\therefore\ ABC=-11$.

$\therefore\ A,B,C$ are the roots of the equation $x^3-x^2+12x-11=0$, which is the equation required.

36. Proposed by O. W. ANTHONY, Mexico, Missouri.

From two points, one on each of the opposite sides of a parallelogram, lines are drawn to the opposite vertics. Through the points of intersection of these lines a line is drawn. Prove that it divides the parallelogram into two equal parts.

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics in the University of Mississippi, University P. O., Mississippi.

Let $OABC$ be the parallelogram, O the lower left-hand vertex, OA the base, D and E points on OA and CB respectively.

If $OA=b$, $OC=a$, $OD=n$, $CE=m$, then, taking OA as the X-axis and OC as the Y-axis, the points O, A, B, C, D, and E will be given by the co-ordinates $(0,0), (b,0), b,a), (0,a), (n,0)$, and (m,a) respectively.

It follows that the equation of OE is $\frac{y}{a}=\frac{x}{m}$;............(1).

The equation of CD is $\frac{y}{a}+\frac{x}{n}=1$;..........(2).

The equation of BD is $y=\frac{-a}{n-b}(x-n)$.....(3).

The equation of EA is $y=\frac{-a}{b-m}(x-b)$......(4).

From (1) and (2) the intersection of OE and CD is

$\left(\frac{mn}{m+n}, \frac{an}{m+n}\right)$ which denote by (x', y').

From (3) and (4) the intersection of BD and EA is

$\left(\frac{b^2-mn}{2b-m-n}, \frac{-a(n-b)}{2b-m-n}\right)$ which denote by (x'', y'').

The equation of the line passing through these points of intersection is

$y-y'=\frac{y'-y''}{x'-x''}(x-x')$....(5).

If the center of the parallelogram is on this line its co-ordinates, $\left(\frac{b}{2}, \frac{a}{2}\right)$, will satisfy (5).

Substituting and reducing, $\frac{1}{2}-\frac{n}{m+n}=\frac{m-n}{2(m+n)}$, or (5) is satisfied.

Since every line which passses through the center of aparallelogram divides it into two equal parts, the proposition is established.

PROBLEMS.

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39. Proposed by J. K. ELLWOOD, Principal of Colfax Schools, Pittsburg, Pennsylvania.

If on the three sides of any plane triangle equilateral triangles be described, the lines joining the centres of these equilateral triangles form an equilateral triangle.

40. Proposed by J. C. CORBIN, Pine Bluff, Arkansas.

If R, r, r_1, r_2, and r_3 be, respectively, the radii of the circumscribed, inscribed, and escribed circles of a $\triangle$, prove $r_1+r_2+r_3-r=4R$.

41. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

Find the length (x) of a rectangular parallelopiped $b=5$ ft. and $h=3$ ft., which can be *diagonally inscribed* in a similar parallelopiped $L=83$ ft., $B=64$ ft., and $H=50$ ft.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

26. **Proposed by Professor J. F. W. SCHEFFER, M. A., Hagerstown, Maryland.**

According to Bessel, the ratio of the squares of the polar diameter of the earth to that of the equatorial diameter, is .9933254. Find what *latitude* the angle made by a body falling to the earth, with a perpendicular to the surface, is greatest. Find, also, this maximum angle.

Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Let ϕ=the required geographical latitude, and ϕ'=the geocentric latitude of the same place; then *Chauvenet's Spherical and Practical Astronomy*, Vol. I., p. 98, we deduce $\phi'=\tan^{-1}[(b^2/a^2)\tan\phi]=\tan^{-1}[(1-e^2)\tan\phi]$.

$\therefore \quad (\phi-\phi')=\phi-\tan^{-1}[(1-e^2)\tan\phi],$=a Maximum.

$$\therefore \quad \frac{d(\phi-\phi')}{d\phi}=1-\frac{(1-e^2)(1+\tan^2\phi)}{1+(1+e^2)^2\tan^2\phi}=0.$$

$$\therefore \quad \phi=\tan^{-1}\left[\sqrt{\left(\frac{1}{1-e^2}\right)}\right]=\tan^{-1}(1.0033541)=45^\circ\ 5'\ 45''.32,$$

and $\phi'=\tan^{-1}(.9966571)=44^\circ\ 54'\ 14''.67$.

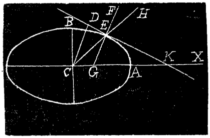
Hence $(\phi-\phi')=11'\ 30''.65$; and this result is found in the already-named Manual of Astronomy, Vol. II., third Table, p. 577.

II. **Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.**

Not taking into account the eastward deviation due to the rotation of the earth we can proceed as follows:

Let HEC be the direction the body falls, FEG the perpendicular to the earth's surface at E, DEK the tangent to the meridian at E, $CA=a$,

$CB=b$, $\angle ECA=\theta$, $\angle DCE=\angle CEG=\phi$, $\angle EKX=\beta$, co-ordinates of $E=(x,y)$.



Then $\tan\beta=-\dfrac{b^2x}{a^2y}$, $\tan\theta=\dfrac{y}{x}$, also

$\beta=90^\circ+\theta+\phi$, $\tan\beta=\tan(90^\circ+\theta+\phi)$

$=-\cot(\theta+\phi)$.

$$\therefore\ \frac{b^2x}{a^2y}=\cot(\theta+\phi)=\frac{\cot\theta\cot\phi-1}{\cot\theta+\cot\phi}$$

$$=\frac{\frac{x}{y}\cot\phi-1}{\frac{x}{y}+\cot\phi}.$$

$\therefore\ \tan\phi=\dfrac{a^2-b^2}{a^2b^2}xy=\text{maximum}\ldots.(1).\quad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\ldots.(2).$

The first differentials of (1) and (2) give $b^2x^2=a^2y^2$.

$\therefore\ x=\dfrac{a}{\sqrt{2}},\ y=\dfrac{b}{\sqrt{2}}$. $\therefore\ \tan\theta=\dfrac{b}{a}=(.9933254)^{\frac{1}{2}}=.9966659$. $\therefore\ \theta=44^\circ\ 54'\ 14''.9$ = the latitude $\tan\phi=\dfrac{a^2-b^2}{2ab}=\dfrac{.0066746}{1.993318}=.003348$. $\therefore\ \phi=1'\ 30''.5$ = maximum angle made with the perpendicular.

Also solved by Professor *C. W. M. Black*, and the *Proposer*.

PROBLEMS.

36. Proposed by H. C. WHITAKER, B. Sc., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

A cube is revolved on its diagonal as an axis. Define the figure described and calculate its volume.

37. Proposed by J. A. CALDERHEAD, Superintendent of Schools, Limaville, Ohio.

A man ties two mules one to the outside of a circular wall, the other to the inside. If the lengths of the ropes of each is one-fourth the circumference of the wall, and both together can graze over one acre of ground: find the circumference of the wall.

MECHANICS.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

14. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University Post Office.

"The center of a sphere of radius c moves in a circle of radius a and generates thereby a solid ring, as an anchor ring; prove that the moment of inertia of this ring about an axis passing through the center of the direct circle and perpendicular to a plane is $\frac{\pi^2\rho a c^2}{4}(4a^2+3c^2)$."

IV. Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let E be the centre of the sphere, AB, the axis of revolution, be the axis of x, F any point in area of the circle, $DF=y$, $HF=y'$, $CE=a$, $GE=c$.

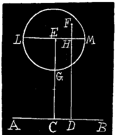
Let dA be the element of area, then the volume of the elementary ring generated by dA is $2\pi y dA$, and its mass, $2\pi\rho y dA$.

$\therefore$ the moment of inertia of this elementary ring relative to the axis of x, is $2\pi\rho y^3 dA$.

$\therefore$ the moment of inertia required $=I=2\pi\rho\Sigma y^3 dA$.

$I=2\pi\rho\Sigma(a+y')^3 dA$, since $y=a+y'$,

$=2\pi\rho\Sigma(a^3+3a^2y'+3ay'^2+y'^3)dA$.



But the curve is symmetrical with respect to the axis LM, $\therefore$ $\Sigma y'dA=0$, $\Sigma y'^3 dA=0$, and by definition, $\Sigma y'^2 dA=Ak^2=\pi c^2k^2$. $\therefore$ $I=2\pi\rho a\times\pi c^2(a^2+3k^2)$

$=2\pi^2\rho ac^2(a^2+3k^2)$; but $k^2=$radius of gyration$=\frac{c^2}{4}$.

$\therefore$ $I=2\pi^2\rho ac^2(a^2+\frac{3}{4}c^2)=\frac{\pi^2\rho ac^2}{2}(4a^2+3c^2)$.

V. Solution by P. H. PHILBRICK, M. Sc., C. E., Lake Charles, Louisiana.

$OP=a=$the radius of directing circle and $Pa=c=$radius of the sphere.

Take tangent Px for the axis of x. Let $aPx=\theta$, $aPc=d\theta$. Draw abR and cdN perpendicular to Px. Take m at middle of ab. Now $am=c\cos\theta$,

$\therefore$ $aR=a+c\cos\theta$ and $bR=a-c\cos\theta$. Thickness of $abcd=c\cos\theta d\theta$.

Area of circle, radius aR, $=\pi(a+c\cos\theta)^2$.

$\therefore$ vol. of circular lamina $aR=\pi(a+c\cos\theta)^2\, c\cos\theta d\theta$ and moment of Inertia about axis through O perpendicular to plane

$$\cdot\ OPx=I=\frac{\pi}{2}(a+c\cos\theta)^4\, c\cos\theta d\theta.$$

Similarly, moment of Inertia of lamina $bR=I'$

$$=\frac{\pi}{2}(a-c\cos\theta)^4\, c\cos\theta d\theta.$$

$\therefore$ moment of Inertia of lamina $ab=I-I'=I''=4\pi ac^2(a^2\cos^2\theta d\theta+c^2\cos^4\theta d\theta)$.

$$\text{But} \int_0^\pi \cos^2\theta d\theta=\frac{\pi}{2};\ \text{and} \int_0^\pi \cos^4\theta d\theta=\tfrac{3}{4}\int_0^\pi \cos^2\theta d\theta=\frac{3\pi}{8}.$$

$\therefore\ I''=\dfrac{\pi^2 ac^2}{4}(8a^2+6c^2)^4$. The moment of Inertia about an axis through O and in the plane OPx is of course one half of the above amount, or $\dfrac{\pi^2 ac^2}{4}(4a^2+3c^2)$.

[NOTE.—An excellent solution of this problem was received from Professor Wiggins, Richmond, Indiana, but as it has been lost we are unable to publish it.—Editor.]

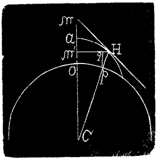
15. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Show that the *eastward deviation* of bodies falling from a great height is

$$E_d=\frac{4\pi t(H-\frac{1}{2}\triangle)\cos\phi}{3T}.$$

Solution by the PROPOSER.

Let r=the radius of the earth, T=86164.2=the *mean-time* seconds in a siderial day, ϕ=the observer's latitude, $(r+H)$=the distance from the center of the earth to the point from which the body fell, and t=the time of the body's motion in seconds: then the *horizontal velocity* of the falling body in the direction of the tangent to the circle of latitude at the place of observation, is



$V=\dfrac{2\pi(r+H)\cos\phi}{T}$ ....(1); and this velocity is caused by the rotation of the earth. Obviously the *horizontal space* described by the body in the time t, is $S_1=Vt=\dfrac{2\pi t(r+H)\cos\phi}{T}$ ....(2).

The point on the earth and vertically below the point from which the body fell, passes during this time t through an arc

$$S_2=\frac{2\pi tr\cos\phi}{T}\ ....(3).$$

Hence, the eastward deviation of the falling body, if *the direction of gravity remained parallel toward this body*, would become

$E_d = S_1 - S_2 = \frac{2\pi t H\cos\phi}{T}$ .... (4); although this *change of direction* is not great, yet it must not by any means be neglected—even though the variation in the *intensity* of gravity is frequently neglected. Suppose the body in its descent to have arrived at H, and let $m'H = x$. Let w represent the body's velocity in the direction $m'H$, and v that in the direction HC; then $dv = gdt$ .... (5).

Considering $HC = r$, we have the proportion,

$$-dw : dv :: Hq : Hp :: m'H : HC :: x : r.$$

$\therefore\ dw = -\frac{x}{r}dv, = -\frac{gx}{r}dt$ .... (6), the negative sign showing that w decreases as r *increases*. Since $w = dx / dt$, (6) becomes

$$dw = -\frac{gxdx}{rw}.\quad \therefore\ wdw = -\frac{g}{r}xdx \ldots (7).$$

Integrating (7) and remembering that when $x=0$ the initial value of w is V as given in (1), we have $w^2 = V^2 - \frac{gx^2}{r}$. $\therefore\ w = V\sqrt{\left(1-\frac{gx^2}{rV^2}\right)}$ .... (8).

With $w = dx / dt$, we now easily deduce

$Vdt = \frac{dx}{\sqrt{\left(1-\frac{gx^2}{rV^2}\right)}} = \left(1+\frac{1}{2}\cdot\frac{gx^2}{rV^2} + \text{higher powers of } x\text{, which may be neglected.}\right)dx$ .... (9).

Integrating (9), observing that $C=0$ since $x=0$ when $t=0$, and without *appreciable* error putting $x^3 = (Vt)^3$, we have the expression

$$x = Vt - \frac{1}{6}\cdot\frac{gVt^3}{r} \ldots (10).$$

If $\triangle$ represents the *excess of descent* in vacuo above that in air, we have

$$\frac{1}{2}gt^2 = H + \triangle \ldots (11). \quad \therefore\ x = Vt - \frac{(H+\triangle)Vt}{3r} \ldots (12).$$

Transforming (12) by means of (1) and without *sensible* error writing *unity* for $(r+H) / r$, we have $x = \frac{2\pi t[(r+H)-\frac{1}{3}(H+\triangle)]\cos\phi}{T}$ .... (13).

For obvious reasons subtracting (3) from (13), etc., we obtain

$$E_d = \frac{4\pi t(H-\frac{1}{2}\triangle)\cos\phi}{3T} \ldots (14),$$

which is the *expression* for the eastward deviation of bodies falling from a great height, given in *Young's General Astronomy*.

Also solved by *G. B. M. Zerr*. His solution will appear next month.

PROBLEMS.

22. **Proposed by DE VOLSON WOOD, C. E., Professor of Mechanical and Electrical Engineering in Stevens Institute of Technology, Hoboken, New Jersey.**

A prismatic bar having a uniform angular velocity ω and a linear velocity of v feet per second, suddenly snaps (by the disappearance of the cohesive force) into an indefinite number of equal parts: required the resultant angular velocity of each piece and the locus of the parts at the end of t seconds after rupture.

23. **Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University Post Office., Mississippi.**

A heavy particle is placed upon the surface $\sqrt{x}+\sqrt{y}+\sqrt{z}=c$. The axis of z being vertical and the coefficient of friction being $1/\sqrt{2}$, show that a point of equilibrium (all friction possible being brought into action) z is a harmonical mean between x and y.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

13. **Proposed by CHARLES E. MYERS, Canton, Ohio.**

A soap bubble 2 inches in diameter, is filled with one part of hydrogen gas and 15 parts of air. If the bubble just floats in the air, find the thickness of the film.

I. **Solution by H. C. WHITAKER, B. S., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.**

Take, as the unit of weight, the weight of air filling a sphere of one inch radius, then the weight of a corresponding volume of hydrogen is .06927 and of a corresponding volume of water is 792.24. Take x=inner radius of sphere.

The weight of the hydrogen is $\frac{.06927}{16}x^3$;

The weight of the air in the bubble is $\frac{15}{16}x^3$;

The weight of the water is $792.24(1-x^3)$;

The weight of the air displaced is 1.

Making the sum of the first three of these equal to the last and solving, $x=.999977$, and the required thickness is .000023 inches.

II. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

In solving this problem many things should be considered. We should know the temperature, the barometric pressure and the dew point in order to calculate the amount of aqueous vapor present in the air.

Not knowing the above it will be well to proceed as follows:

Let r=outside radius of bubble, x=inside radius.

ρ=density of air
ρ_1=density of hydrogen,
ρ_2=density of soap film,
all at the given temperature and pressure and all referred to normal air as standard.

$$\text{Then } \tfrac{4}{3}\pi(r^3-x^3)=\text{volume of soap film,}$$

$$\tfrac{4}{3}\pi\times\frac{r^3}{16}=\text{volume of hydrogen,}$$

$$\tfrac{4}{3}\pi\times\frac{15r^3}{16}=\text{volume of air.}$$

$$\therefore\quad \tfrac{4}{3}\pi(r^3-r^3)\rho_2+-\tfrac{4}{3}\pi\times\frac{r^3}{16}\rho_1+\tfrac{4}{3}\pi\times\frac{15r^3}{16}\rho=\tfrac{4}{3}\pi r^3\rho,$$

$$\therefore\quad 16r^3\rho_2-16x^3\rho_2+x^3\rho_1+15c^3\rho=16r^3\rho,$$

$$\therefore\quad x^3=\frac{16r^3(\rho_2-\rho)}{16\rho_2-15\rho-\rho_1},\quad\therefore\quad x=2r\sqrt[3]{\left(\frac{2(\rho_2-\rho)}{16\rho_2-15\rho-\rho_1}\right)}.$$

$$r-x=r-2r\sqrt[3]{\left(\frac{2(\rho_2-\rho)}{16\rho_2-15\rho-\rho_1}\right)}=\text{required thickness.}$$

Let the conditions be normal and suppose ρ_2, referred to water, is 1.1. Then since air, referred to water, is .001293, ρ_2, referred to air, is 850 73473, $\rho=1$, $\rho_1=.0693$, $r=1$.

$$\text{Then } r-x=1-2\sqrt[3]{\left(\tfrac{1199.46946}{13596.68638}\right)}=.000023.$$

Also solved by *P S. Berg*, *F. P. Matz*, and the *Proposer*.

14. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee

I have a glass paper-weight in the form of a regular icosahedron. I let the sun's rays fall upon it, at various angles, also upon one of the vertices. How many complete spectra will be formed? How many will be of white light? What position will give maximum number of spectra?

[No solution to this problem has as yet been furnished by our contributors, and I see no way of solving it. If a solution is possible it will be a very pretty one. —EDITOR.]

PROBLEMS.

22. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

From what kind of dry wood must a ship's log be cut, in order that the log may float with its center of gravity at the water's surface?

PROBLEMS.

23. Proposed by G. B. M. ZERR, A. M., Principal High School, Staunton, Virginia.

Pliny says, "Thales determined the cosmical setting of the Pleiades to have happened in his time 25 days after the vernal equinox." Determine the time when Thales lived from the following data:—Latitude of Miletus $37^\circ 30'$, the precession of the equinox $50''.34$ annually, the R. A. of Alcyon (η Tauris) Jan. 1, 1895, 3h. 41m. 15sec. declination $23^\circ 46' 49''$ N.

24. Proposed by D. H. DAVISON, Minonk, Illinois.

For the purpose of locating the most eligible point for a county seat, it is required to determine the center of a county whose dimensions are as follows: Beginning at the S. W. corner. Thence east 15 miles, thence N. 33$\frac{3}{4}$ miles, thence W. 6 miles to north end of a meridian line running south through the county: thence southwesterly to a point being 6 miles west from the meridian line and 9$\frac{3}{4}$ miles south of its north end. Thence S. 3 miles, thence W. 3 miles, thence S. 21 miles to place of beginning.

[NOTE.—A solution to problem 12 will appear in March number. EDITOR.]

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

DR. HALSTED'S LATEST TRANSLATION.

By LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago, Chicago, Illinois.

It is a matter worthy of remark that, already master of four modern and two dead languages and a translator from several of them of no little repute, Dr. Halsted has vigorously attacked the Russian language with its 36 strange hieroglyphics and now gives to the world an insight into the best scientific thought of Russia, in the shape of a translation of the address by Prof. Vasiliev, President of the Physico-Mathematical Society of Kasan, pronounced last year at the meeting of the Imperial University of Kasan in commemoration of their illustrious compatriot Lobachevsky.

From every one devoted to Mathematics or Philosophy, or indeed to the highest advance of human thought in any form, this address will call forth the deepest admiration for Lobachevsky, now recognized as one of the greatest

intellectual revolutionizers the world has ever had. It will arouse a deeper enthusiasm for scientific achievement and widen the horizon of every reader.

Surely no mathematician should miss this gem from fartherest Russia, which, thanks to the rare enthusiasm and energy of Professor Halsted, is easily accessible to all.

A REPLY TO PROFESSOR WHITAKER.

By H. W. DRAUGHON, Olio, Mississippi.

Professor Whitaker, in his reply, devotes a great part of his space to attacking positions which I have never occupied. I will, therefore, consider those points only, in his article, which bear on the subject under discussion. In regard to the expression, $3+\sqrt{2}$, I will state that the sign before $\sqrt{2}$ does not indicate that the positive value of $\sqrt{2}$ is to be taken.

Professor Whitaker does not deny that I find the value of x^2 correctly from the equation, $x^2+2x=3$; he only claims that this equation is not similar to the equation $\sqrt{}(x+4)-\sqrt{}(x-4)=4$. In proof of this, he asserts that, the product of the equation, $x^2+2x-3=0$,and the equations formed by "changing signs" is of the 8th degree. This is not true, when we consider x^2, whose value is required, as the unknown quantity.

I beg leave to remind Professor Whitaker that the performance of this operation on L. B's equation gives an equation of the 2nd degree with reference to $\sqrt{}(x+4)$ or $\sqrt{}(x-4)$. In my example, the produce, is of the 4th degree with reference to x^2. So it appears that the dissimilarity is not great enough to consider, after all.

Let us take a simple equation; $2+x=0\ldots.(1)$, for instance. We readily find $x^2=4\ldots.(2)$. Now Professor Whitaker claims that the value of x is essentially positive; $\therefore$ from (2) $x=+2$. This value fails to prove when substituted in (1). Let us now multiply as Professor Whitaker suggests, then we have, $(2+x)(2-x)=4-x^2=0$, an equation of the first degree with reference to x^2.

We have here, therefore, an equation which has very much less than "a quarter of a chance of having one root",-if we preclude as Professor Whitaker does, negative values of an expression preceded by the sign $+-$, and, at the same time the product of the equation and the equation obtained by "changing signs" is of the first degree.

I will close, by the application of the principle I gave in my former article, to the solution of L. B's equation. We have, $\sqrt{}(x+4)-\sqrt{}(x-4=4\ldots(1)$. Put $x+4=y^2\ldots.(2)$, and $x-4=z^2\ldots.(3)$; then (1) becomes, $y-z=4\ \ldots(4)$. From (2) and (3), we obtain, $y^2-z^2=8\ldots.(5)$. From (4) and (5), we find, $y=3$, and $z=-1$. From (2) or (3) we now find, $x=5$. Now, I do not think

there is any hocus-pocus about this. It makes the question clear and enables us to find out everything about the given equation. I feel sure that this view of the matter is in strict accordance with the principles of mathematics.

WANTED.—Dr. G. B. Halsted, Professor J. N. Lyle, Counsellor Dolman, and all other apostles and post-graduate disciples of Lobatschewsky, to inform the numerous readers of the MONTHLY wherein consists *the difference* between the Euclidian Geometry and the Non-Euclidian Geometry. What is *Ideal Space? Hyper-space? Pseudo-spherical*, as used by Professor Lyle in the November MONTHLY.—READER.

NOTE.—Dr. Artemas Martin pointed out to me in a letter (which I misplaced at the time and only recently recovered) that the expressions $m+\sqrt{(2mn)}, n+\sqrt{(2mn)}, m+n+\sqrt{(2mn)}$, given in my article in the January '94 number of the MONTHLY, for the sides of a right triangle, can be reduced to Maseres' expressions, p^2-q^2, $2pq$, p^2+q^2, respectively, by substituting $(p-q)^2$ for m and $2q^2$ for n. I wish to thank Dr. Martin.

LEONARD E. DICKSON, M. A.

EDITORIALS.

THIS number of the MONTHLY was mailed February 28th. It has been cut short, but our readers may look for a good, full number in March.

MARCH Number will be mailed between the 20th and 25th of the month. If you do not get your copy soon after the 25th write to the publishers at once.

IF any of our subscribers have not received any one of the 14 numbers of the MONTHLY already issued write the publishers, and if it is possible, the missing copy will be sent.

AT THE last meeting of the American Mathematical Society, Dr. Macfarlane read a paper on the *Principles of Differentiation in Space-analysis*, which contains among other results, the true generalization for space of Taylor's Theorem. Dr. Macfarlane says, there are many indications pointing to this as the coming subject.

OUR readers will be disappointed because of the absence of the portrait of Professor Chauvenet. No pains were spared on our part in trying to obtain a plate, but our efforts were futile. We may be able to secure a portrait before the end of the year and if we do, will send each of our subscribers a copy, so that it may be bound up with the year's volume.

ERRATA IN PROFESSOR DICKSON'S ARTICLE: p. 9, l. 11 read s=or$<(m-1)$ 12; l. 11 read $--A_{m-s}$: p.40, ls. 34 and 36 for $\pm A_{p-1}$ read $\mp$: l. 35, for $\pm A_{p-2}$ and $\pm A_p$ read $\mp$; l. 37, for $A_1+2=0$ read not=0.